Defect Relaxation and Coarsening Exponents

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The coarsening exponents describing the growth of long-range order in systems quenched from a disordered to an ordered phase are discussed in terms of the decay rate $\omega(k)$ for the relaxation of a distortion of wavevector k applied to a topological defect. For systems described by order parameters with $Z(2)$ ('Ising') and $O(2)$ ('XY') symmetry the appropriate defects are domain walls and vortex lines respectively. From $\omega_k \sim k^z$, we infer $L(t) \sim t^{1/z}$ for the coarsening scale, with the assumption that defect relaxation provides the dominant coarsening mechanism. The $O(2)$ case requires careful discussion due to infrared divergences associated with the far field of a vortex line. Conserved, non-conserved, and 'intermediate' dynamics are considered, with either short-range or long-range interactions. In all cases the results agree with an earlier 'energy scaling' analysis.

I. INTRODUCTION

Coarsening is a characteristic property of a system quenched into an ordered phase from a disordered phase, and describes the establishment of long-range order over ever larger length scales as time increases [1]. We concentrate here on systems which exhibit the property of 'dynamical scaling', in which the pair correlation function $C(r,t) \equiv \langle \phi(\mathbf{x},t)\,\phi(\mathbf{x}+\mathbf{r},t)\rangle$ of the order-parameter field $\phi(\mathbf{r},t)$ has the scaling form $C(r,t) = f[r/L(t)]$. The characteristic length scale $L(t)$ generally has a power-law dependence on time, $L(t) \sim t^{1/z}$, sometimes with logarithmic corrections, where z is a kind of dynamic exponent for the coarsening process.

Much effort in recent years has been devoted to the determination of the exponent z (and any additional logarithmic factors) for a range of different models. Although the experimentally most relevant case of a scalar order parameter has been understood for some time, through intuitive arguments based on the scaling assumption, and through exactly solved limits [2], a general approach has been lacking until recently. The 'energy-scaling' approach of Bray and Rutenberg [1,3,4] filled this gap by providing a very general approach based on the role of the topological defects nucleated during the quench. These defects are simple domain walls for a scalar field, vortices/vortex lines for a two-component vector field, and monopoles for three-component vector field (where we are assuming rotational invariance for the last two models, i.e. these models possess $O(2)$ and $O(3)$ symmetry respectively). The energy-scaling method is suitable for any system for which the scaling hypothesis holds and the dynamics are purely dissipative. These include the standard models 'A' and 'B' of the Hohenberg and Halperin classification [5], describing the simplest systems with non-conserved and conserved order parameter respectively. In the context of phase-ordering kinetics, the models are described by the time-dependent Ginzburg-Landau (TDGL) equation (see section II) and the Cahn-Hilliard equation (section III).

Powerful though the energy-scaling method is, the arguments are not completely straightforward and some care is required in its application (see [1] for a detailed discussion). In this paper, therefore, we will discuss a much 'cleaner' approach that addresses the dynamics of the topological defects in a more direct way. Specifically we consider a single plane domain wall, or straight vortex line, and apply a periodic perturbation of wavevector k. The perturbation then relaxes away, with asymptotic relaxation rate $\omega(k)$. A central assumption, as before, is the validity of the scaling hypothesis. If there is a single characteristic length scale $L(t)$, then the relaxation dynamics of the defect should be reflected in the coarsening dynamics, through $\dot{L}/L \sim \omega(k)$ with $k \sim 1/L$. So if $\omega(k) \sim k^z$ for $k \to 0$, we expect $L(t) \sim t^{1/z}$. More simply, one can set $\omega \sim 1/t$ and $k \sim 1/L$ to obtain the same result.

This approach has been used before for systems with domain walls, and the conventional $z = 2$ and 3 for systems with conserved and nonconserved order parameter respectively have been recovered [6–8]. Here we generalize to systems with 'intermediate' dynamics (to be defined below), and/or long-range interactions, and extend the method to systems with $O(2)$ symmetry. In this way we can confront most of the predictions of the energy scaling method. Since the present approach requires extended defects, however, it cannot treat the $O(2)$ system in $d = 2$ spatial dimensions, or the $O(3)$ system in $d = 3$, for which the relevant topological defects (vortices and monopoles respectively) are points. We therefore restrict our attention to scalar systems in $d = 2, 3$ and $O(2)$ systems in $d = 3$. For these cases, all our results agree with the energy scaling predictions, although some subtleties arise in the case of the $O(2)$ system due to infrared singularities associated with the far field of the vortex line.

The paper is organized as follows. Section II deals with nonconserved dynamics ('model A'). The method is introduced in the simple context of domain walls. The extension to vortex lines is then discussed, followed by the generalization to long-range interactions. Section

III deals with conserved ('model B') and 'partially conserved' dynamics, for domain walls and vortex lines with both and short-range interactions. Detailed comparisons with the predictions of the energy scaling method are made at each stage. In particular, the energy scaling prediction $L(t) \sim (t \ln t)^{1/4}$ for the conserved $O(2)$ model in $d = 3$ (or, more generally, $d \geq 3$) is recovered. The paper concludes with a discussion and summary.

II. NONCONSERVED DYNAMICS

A. Domain walls

The simplest exemplar of the defect relaxation approach is the dynamics of an interface (domain wall) separating two equilibrium phases. The time-dependent Ginzburg Landau (TDGL) equation is a continuum dynamical model for such a process:

$$\partial_t \phi = \nabla^2 \phi - V'(\phi) \ , \tag{1}$$

where $\phi(\mathbf{x}, t)$ is the scalar order-parameter field and $V(\phi)$ is a symmetric double-well potential, e.g. $V(\phi) = (1 - \phi^2)^2/4$, whose minima $\phi = \pm 1$ represent the two bulk phases.

Consider first a single planar interface separating the two phases. The normal to the interface define the z-direction. The order parameter depends only on z, and satisfies the time-independent version of (1), $d^2\phi_0/dz^2 = V'(\phi_0)$, with boundary conditions $\phi_0(\pm\infty) = \pm 1$. We now impose a small periodic perturbation to the interface in the x-direction with wavevector k

$$\phi(x, z, t) = \phi_0(z) + A\phi_1(z) \exp(ikx - \omega_k t) \ , \tag{2}$$

where the amplitude A is small. The relaxation rate ω_k defines the timescale, $\tau_k = 1/\omega_k$, for the relaxation of a perturbation with characteristic length-scale $2\pi/k$. Substituting this form into (1), and linearizing in A, gives the eigenvalue equation $H\phi_1 = \omega_k \phi_1$, with 'Hamiltonian'

$$H = -\frac{d^2}{dz^2} + k^2 + V''(\phi_0) \ . \tag{3}$$

This is conveniently thought of as a quantum mechanical Hamiltonian operator. Note that $V''(\phi_0)$ is positive at $z = \pm\infty$ (where $\phi_0 = \pm 1$), and negative at $z = 0$ (where $\phi_0 = 0$). For the specific case $V(\phi) = (1-\phi^2)^2/4$ one has $V''(\phi_0) = 3\phi_0^2(z) - 1$, which equals -1 at $z = 0$ and tends to $+2$ for $z \to \pm\infty$. It follows that $V''(\phi_0)$ represents a potential well which must have at least one bound state. In fact, since $k = 0$ corresponds to a uniform translation of the interface, we know that $\omega_{k=0} = 0$, with eigenfunction $\phi_1 = d\phi_0/dz$. Also, since this function has no nodes, it must be the ground state. Since the k-dependence of H is simply the additive constant k^2 in this simple model, it follows that $d\phi_0/dz$ is the ground state for all k, with eigenvalue

$$\omega_k = k^2. \tag{4}$$

Higher eigenvalues are separated by a gap from the ground state, so any component of the corresponding eigenfunctions, in the initial displacement of the interface, relaxes quickly to zero. For this reason, we concentrate here only on the groundstate eigenfunction, which is the 'slow mode'. Neglecting any contribution from the other eigenfunctions, (2) can be written, correct to leading order in the amplitude A, as $\phi(z, x, t) = \phi_0(z + A\exp[ikx - \omega_k t])$, corresponding simply to a modulated interface located at $z(x, t) = -A\exp[ikx - \omega_k t]$. Note that this simple interpretation is a consequence of $\phi = d\phi_0/dz$, which will not be true in general.

Equation (4) has the form $\omega_k = k^z$ with $z = 2$, from which infer the coarsening growth law $L(t) \sim t^{1/z} = t^{1/2}$ for this, the simplest of our dynamical models.

B. Vortex Lines

If the order-parameter field is the two-component vector field $\vec{\phi}(\mathbf{x}, t)$ of the $O(2)$ model, the topological defects are vortex lines. The analogue of (1) for this system is

$$\partial_t \vec{\phi} = \nabla^2 \vec{\phi} - dV/d\vec{\phi} \ , \tag{5}$$

where the 'wine-bottle' potential $V(\vec{\phi})$ is a function of $|\vec{\phi}|$ only. For definiteness, we will use the form $V(\vec{\phi}) = (1 - |\vec{\phi}|^2)^2/4$, with minima on the ground-state manifold $|\vec{\phi}| = 1$, although the conclusions are completely general. For this potential (5) becomes

$$\partial_t \vec{\phi} = \nabla^2 \vec{\phi} + (1 - |\vec{\phi}|^2)\, \vec{\phi} \ . \tag{6}$$

This equation has a time-independent solution $\vec{\phi}_0(\mathbf{r})$, corresponding to a vortex line in the z direction, where we have introduced the coordinate system $\mathbf{x} = (\mathbf{r}, z)$ of coordinates normal and parallel to the vortex. This is a solution of (6) satisfying the boundary conditions $\vec{\phi}_0(0) = 0$, $\vec{\phi}_0(\mathbf{r}) \to \hat{\mathbf{r}}$ for $|\mathbf{r}| \to \infty$, where $\hat{\mathbf{r}}$ is a unit vector.

By analogy with (2) we now perturb the vortex line by adding a periodic modulation with wavevector k in the form

$$\vec{\phi}(\mathbf{r}, z, t) = \vec{\phi}_0(\mathbf{r}) + A\vec{\phi}_1(\mathbf{r}) \exp(ikz - \omega_k t) \ . \tag{7}$$

Substituting in (6), and linearizing in A, gives the eigenvalue equation

$$(\nabla_r^2 - k^2)\vec{\phi}_1 + (1 - |\vec{\phi}_0|^2)\, \vec{\phi}_1 - 2(\vec{\phi}_1 \cdot \vec{\phi}_0)\, \vec{\phi}_0 = -\omega_k \vec{\phi}_1 \ , \tag{8}$$

where ∇_r^2 is the Laplacian operator in the $\mathbf{r}$-plane. From the same physical considerations that we used for domain walls, we expect a null eigenfunction at $k = 0$, corresponding to a uniform displacement of the vortex line

transverse to its length. Since $\vec{\phi}_0(\mathbf{r}+\mathbf{a}) = \vec{\phi}_0 + \mathbf{a}\cdot\nabla_r\vec{\phi}_0$, to first order in $\mathbf{a}$, we identify a family of null eigenfunctions $\vec{\phi}_1 = \mathbf{a}\cdot\nabla_r\vec{\phi}_0$, parametrized by the direction of $\mathbf{a}$. For convenience we may choose a basis set of two such eigenfunctions corresponding to $\mathbf{a}$ being a unit vector along the x and y axes (where $\mathbf{r} = (x, y)$) respectively. The two basis functions are then $\partial_x\vec{\phi}_0$ and $\partial_y\vec{\phi}_0$. They are orthogonal by symmetry. Substitution into (8) confirms that these are null eigenfunctions, i.e. $\omega_{\mathbf{k}=\mathbf{0}} = 0$. Since the k^2 term in (8) can simply be absorbed into ω_k, it follows that, just as for domain walls, the smallest eigenvalue for any k is exactly given by

$$\omega_k = k^2 \ . \tag{9}$$

We conclude that the dynamical exponent is again $z = 2$. This agrees with the energy-scaling result.

For future reference we note that the eigenfunctions $\partial_x\vec{\phi}_0$ and $\partial_y\vec{\phi}_0$ are not normalizable in an infinite two-dimensional space, since the normalization integrals are logarithmically divergent (see below). When the normalization is important, we will take the vortex line to lie along the symmetry axis of an infinitely long cylinder of radius R.

C. Long-Range Interactions: Domain Walls

The above approach may readily be generalized to the case where the underlying interactions are long-ranged in space. Consider, for example, a ferromagnetic model, in which the exchange interaction $J(|\mathbf{r}-\mathbf{r}'|)$ falls off with distance as $|\mathbf{r}-\mathbf{r}'|^{-(d+\sigma)}$ for large $|\mathbf{r}-\mathbf{r}'|$. The long-range character of these interactions is typically only 'relevant' (i.e. affects the scaling behavior) if σ is smaller than a critical value σ_c [9]. In the present context, we will find that $\sigma_c \leq 2$, so we will consider only the case $\sigma < 2$ here. For this case, the term in the Ginzburg-Landau free-energy functional generated by long-range interactions has the Fourier-space form $F_{LR} = (g/2)\sum_{\mathbf{k}} k^\sigma \phi_{\mathbf{k}}\phi_{-\mathbf{k}}$. This can be formally represented in the TDGL dynamics, $\partial_t\phi = -\delta F/\delta\phi$, by replacing the usual Laplacian operator by $-(-\nabla^2)^{\sigma/2}$, to give

$$\partial_t\phi = \nabla^2\phi - g(-\nabla^2)^{\sigma/2}\phi - V'(\phi) \ , \tag{10}$$

instead of (1). Expanding around a flat domain wall using (2), where ϕ_0 is now a stationary solution of the full equation (10), gives the eigenvalue equation

$$(d_z^2 - k^2 - g[k^2 - d_z^2]^{\sigma/2} - V''[\phi_0])\phi_1 = -\omega_k\phi_1 \ , \tag{11}$$

where $d_z \equiv d/dz$.

The function $\phi_1 = d\phi_0/dz$ is again a null eigenfunction for $k = 0$, since $k = 0$ corresponds to a simple translation of the interface. The term in k^2 in (11) simply shifts the eigenvalue by k^2, as before. However, the long-range part of (11) modifies the eigenfunction. We can, nevertheless, compute the desired small-k behavior of the ground-state eigenvalue perturbatively, using the unperturbed ground state eigenfunction $d\phi_0/dz \equiv \phi_0'(z)$:

$$\omega_k = k^2 + g\,\frac{\int_{-\infty}^{\infty} dz\,\phi_0'(z)\,([k^2 - d_z^2]^{\sigma/2} - [-d_z^2]^{\sigma/2})\,\phi_0'(z)}{\int_{-\infty}^{\infty} dz\,[\phi_0'(z)]^2} \ . \tag{12}$$

The integral in the numerator is conveniently evaluated in Fourier space. The function $\phi_0'(z)$ is sharply peaked at the interface, with a width of order ξ, the interfacial thickness, and a peak height of order $1/\xi$. Its Fourier transform, therefore, is very broad (with width $1/\xi$) and for $q\xi \ll 1$ is equal to $\phi_0(\infty) - \phi_0(-\infty) = 2$. The integral in the numerator in (12) becomes, therefore

$$4\int_{-\infty}^{\infty}\frac{dq}{2\pi}([k^2 + q^2]^{\sigma/2} - |q|^\sigma),$$

for small k, provided the integral converges. The latter condition requires $\sigma < 1$. In this regime, the important values of q in the integral are of order k, so our replacement of $\phi_0'(z)$ by $2\delta(z)$ is justified for $k\xi \ll 1$. The integral is then easily evaluated to give $k^{1+\sigma}$ (up to constants) for $\sigma < 1$.

For $\sigma > 1$, the replacement of ϕ_0' by a delta function is no longer valid, since the integral in the numerator in (12) would not converge. The Fourier transform of ϕ_0' falls off at $q\xi \approx 1$, however, to converge the integral. In this case ($\sigma > 1$) the characteristic value of q is of order $1/\xi$, instead of k, and we can expand the integrand up to order k^2 for $k\xi \ll 1$. The numerator then becomes of order $k^2\xi^{1-\sigma}$ for $\sigma > 1$. A similar line of reasoning for the marginal case $\sigma = 1$ leads to a $k^2\ln(1/k\xi)$ behavior for $k\xi \ll 1$. The simple k^2 dependence for $\sigma > 1$ just renormalizes the amplitude of the leading k^2 term in (12). The integral in the denominator in (12) is a constant of order $1/\xi$.

Putting it all together, retaining only the leading small-k behavior, and discarding constant prefactors, gives the dispersion relation

$$\begin{aligned} \omega_k &\sim k^{1+\sigma}, \quad \sigma < 1, \\ &\sim k^2\,\ln(1/k\xi), \quad \sigma = 1, \\ &\sim k^2, \quad \sigma > 1. \end{aligned} \tag{13}$$

We deduce that the dynamic exponent is $z = 1 + \sigma$ for $\sigma < 1$ and $z = 2$ for $\sigma > 1$. The marginal case $\sigma = 1$ gives (with $k \to 1/L$, $\omega_k \to 1/t$ as usual) the growth law $L \sim (t\ln t)^{1/2}$. All these results are in accord with previous results based on renormalization group [9] and energy scaling [3,4] arguments.

D. Long-Range Interactions: Vortex Lines

The influence of long-range interactions on the dynamics of a vortex line may be discussed in a similar way. Taking, as before, $V(\vec{\phi}) = (1 - |\vec{\phi}|^2)^2/4$, one obtains, analogous to (8) and (11),

$$(\nabla_r^2 - k^2 - g[k^2 - \nabla_r^2]^{\sigma/2} + 1 - |\vec{\phi}_0|^2)\vec{\phi}_1 \\ -2(\vec{\phi}_1 \cdot \vec{\phi}_0)\vec{\phi}_0 = -\omega_k \vec{\phi}_1 \ . \qquad (14)$$

The long-range part can be treated perturbatively for small k as in the scalar case. Since the perturbation is isotropic, the unperturbed eigenfunctions $\partial_x\vec{\phi}_0$ and $\partial_y\vec{\phi}_0$ are not mixed by the perturbation. It follows that, to leading order in perturbation theory,

$$\omega_k = k^2 + g\,\frac{\int d^2r\,\partial_i\phi_{0j}\,([k^2 - \nabla_r^2]^{\sigma/2} - [-\nabla_r^2]^{\sigma/2})\,\partial_i\phi_{0j}}{\int d^2r\,[\partial_i\phi_{0j}]^2} \ , \qquad (15)$$

where the result has been written in a rotationally invariant form, i.e. there are implicit summations over the indices i and j.

We again evaluate the integral in the numerator in Fourier space. In the small-k limit only small wavevector $\mathbf{q}$, corresponding to large $|\mathbf{r}|$, will be important. For large $|\mathbf{r}|$, $\vec{\phi}_0 \to \hat{\mathbf{r}}$, giving $\partial_i\phi_{0j} = (\delta_{ij} - \hat{r}_i\hat{r}_j)/|\mathbf{r}|$. The Fourier transform of this quantity is $(2\pi/q)(\delta_{ij} - \hat{q}_i\hat{q}_j)$. Inserting this in (15) the integral in the numerator becomes, up to constants,

$$\int_{|\mathbf{q}|>1/R} \frac{d^2q}{q^2}([k^2 + q^2]^{\sigma/2} - |\mathbf{q}|^\sigma) \ \sim k^\sigma \ \ln(kR) \qquad (16)$$

for $\sigma < 2$, this condition ensuring the convergence of the integral at large q. The lower cutoff at $q = 1/R$, where we recall that R is the radius of the system in the plane normal to the vortex line, is necessary to regulate the logarithmic singularity at small q.

Now consider the integral in the denominator of (15). Using the large-r form of $\vec{\phi}_0$ everywhere gives the result $\int d^2r/r^2$, i.e. a logarithmically divergent integral. At small r, this can be cut-off at the vortex core size ξ, where the assumed large-r form no longer holds. The large r cut-off is again R. The need for a large-distance cut-off is associated with the non-normalizability of the null eigenfunctions, as discussed in section II B. The upshot is that the denominator is of order $\ln(R/\xi)$, and the leading small-k form of the dispersion relation is

$$\omega_k \sim k^\sigma\,\frac{\ln(kR)}{\ln(R/\xi)}, \qquad \sigma < 2. \qquad (17)$$

Some discussion of this result is in order. If we take the limit $R \to \infty$ in (17), we obtain the well-defined limit $\omega_k \sim k^\sigma$ for a vortex line in an infinite system. We argue, however, that this is not the appropriate limit in which to discuss the implications of (17) for the coarsening dynamics. In a system with many vortex lines, a characteristic scale L can be associated with the line density ρ_V (= length of vortex line per unit volume) via $\rho_V = 1/L^2$. Since the far field of a given vortex line is screened out on this scale, it is L rather than R which is the appropriate cut-off in the coarsening system. Replacing R by L in (17), and making the usual identifications $k \sim 1/L$ and $\omega_k \sim 1/t$, gives the coarsening growth law $L \sim (t/\ln t)^{1/\sigma}$. This agrees again with the predictions of the energy-scaling approach. Similar arguments concerning the nature of the momentum cutoffs will be necessary for the conserved 0(2) model in section III B.

III. CONSERVED DYNAMICS

A. Domain Walls

The standard continuum model for the time evolution of a conserved scalar field is the Cahn-Hilliard equation

$$\partial_t\phi = -\nabla^2[\nabla^2\phi - V'(\phi)] \ . \qquad (18)$$

For present purposes this may be conveniently rewritten in the form

$$-\nabla^2\phi + V'(\phi) + (-\nabla^2)^{-1}\partial_t\phi = 0 \ . \qquad (19)$$

Inserting the form (2) and linearizing in A leads to the eigenvalue equation

$$[k^2\phi_1(z) - \omega_k \int_{-\infty}^{\infty} dz'\,G_k(z - z')\phi_1(z')] \\ +[-d_z^2 + V''(\phi_0)]\phi_1(z) = 0 \ , \qquad (20)$$

where

$$G_k(z - z') = \int_{-\infty}^{\infty} \frac{dq}{2\pi}\,\frac{\exp[iq(z - z')]}{k^2 + q^2} \\ = \frac{\exp(-k|z - z'|)}{2k} \qquad (21)$$

is the Green's function for the operator $(k^2 - d_z^2)$.

As before, $\phi_1 = \phi_0'$ is a null eigenfunction for $k = 0$. For small k the term in the second square bracket in (20), which vanishes for $k = 0$, can be treated perturbatively to give

$$\omega_k = k^2\,\frac{\int_{-\infty}^{\infty} dz\,[\phi_0'(z)]^2}{\int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dz'\,G_k(z - z')\,\phi_0'(z)\,\phi_0'(z')} \ . \qquad (22)$$

The integral in the numerator defines the surface tension, $\gamma = \int_{-\infty}^{\infty} dz\,(\phi_0')^2 \approx 1/\xi$. The function $\phi_0'(z)$ acts like a smeared delta function of width ξ and strength 2. For $k\xi \ll 1$, therefore, we can replace $G_k(z - z')$ by its small-argument limit $1/2k$ in the denominator, to give the result $2/k$. In this limit, therefore,

$$\omega_k = \frac{1}{2}\,\gamma k^3 \ . \tag{23}$$

The dynamic exponent is $z = 3$, in accord with the expected Lifshitz-Slyozov scaling $L \sim t^{1/3}$. A more careful derivation of this result has been given by Shinozaki and Oono [8].

The generalization to 'intermediate' dynamics, or 'non-integer derivative models' [11], in which the leading $-\nabla^2$ in the Cahn-Hilliard equation is replaced by $(-\nabla^2)^{\mu/2}$ (i.e. by $|\mathbf{k}|^\mu$ in Fourier space, instead of k^2) is straightforward. One simply has to insert in (22) the Green's function for $(k^2 - d_z^2)^{\mu/2}$, given by

$$G_k(z - z') = \int_{-\infty}^{\infty} \frac{dq}{2\pi}\, \frac{\exp[iq(z - z')]}{(k^2 + q^2)^{\mu/2}} \ . \tag{24}$$

For $\mu = 2$ our previous result is obtained, while for $\mu = 0$ one gets $G_k(z - z') = \delta(z - z')$, giving the nonconserved result $\omega_k = k^2$. For general μ, (24) gives the scaling form $G_k(z - z') = k^{1-\mu} f(k|z - z'|)$. From the discussion of the case $\mu = 2$ it is clear that we require the result only in the limit $k|z - z'| \ll 1$, i.e. we need the small-argument form of the scaling function $f(x)$. Straightforward analysis gives $f(0) = \text{const.}$ for $\mu > 1$, and $f(x) \sim x^{\mu-1}$ for $\mu < 1$, with $f(x) \sim \ln(1/x)$ for $\mu = 1$. Thus there is a change of behavior at $\mu = 1$. For $\mu > 1$, $G_k(z - z') \sim k^{1-\mu}$ gives (with $\gamma \approx 1/\xi$) $\omega_k \sim \xi^{-1} k^{1+\mu}$. For $\mu < 1$, the double integral $\int dz \int dz' \, |z - z'|^{\mu-1} \phi_0'(z) \phi_0(z')$ is of order $\xi^{\mu-1}$, giving the dispersion relation $\omega_k \sim \xi^{-\mu} k^2$, while for $\mu = 1$ one obtains $\omega_k \sim \xi^{-1} k^2 / \ln(1/k\xi)$. For convenience we summarize these results below, dropping prefactors involving ξ:

$$\begin{aligned} \omega_k &\sim k^{1+\mu}, \quad \mu > 1 \\ &\sim \frac{k^2}{\ln(1/k\xi)}, \quad \mu = 1 \\ &\sim k^2, \quad \mu < 1 \ . \end{aligned} \tag{25}$$

These results have been derived previously by Onuki [11]. The dynamic exponent is $z = 1 + \mu$ for $\mu > 1$ and $z = 2$ for $\mu < 1$. For $\mu < 1$, the exponent is the same as for the nonconserved system: the conservation law is 'irrelevant', in accord with a renormalization group argument [10].

Using $\omega \sim 1/t$ and $k \sim 1/L$ gives the growth laws $L(t) \sim t^{1/(1+\mu)}$ for $\mu > 1$, $(t/\ln t)^{1/2}$ for $\mu = 1$, and the usual nonconserved result $L(t) \sim t^{1/2}$ for $\mu < 1$. These results are again in complete accord with the energy-scaling argument.

Long-range interactions can be included for a conserved order parameter in a straightforward way. Omitting the details, the final result is a rather obvious combination of (12) and (22). For general μ it reads

$$\omega_k = \frac{\int_{-\infty}^{\infty} dz\, \phi_0'(z)\, O_k\, \phi_0'(z)}{\int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dz'\, G_k(z - z')\, \phi_0'(z)\, \phi_0'(z')} \ , \tag{26}$$

where O_k is the operator

$$O_k = k^2 + g[(k^2 - d_z^2)^{\sigma/2} - (-d_z^2)^{\sigma/2}]. \tag{27}$$

Note that the numerator and denominator contain the information about the interactions and the conservation law respectively. It follows that the dynamic exponent is $z(\sigma, \mu) = \min(1 + \sigma, 2) + \max(\mu - 1, 0)$. This clean separation of the role the interactions and the conservation law is mirrored in the 'energy-scaling' approach to calculating growth exponents, where the energy depends only on the interactions (σ) and the energy dissipation rate only on the conservation law (μ).

B. Vortex Lines

The treatment of vortex lines in a conserved 0(2) model follows the same pattern. Consider first the case of 'simple' conservation, $\mu = 2$. The starting point is the Cahn-Hilliard equation for vector fields:

$$(-\nabla^2)^{-1} \partial_t \vec{\phi} = \nabla^2 \vec{\phi} + (1 - |\vec{\phi}|^2)\vec{\phi} \ . \tag{28}$$

Expanding around the stationary vortex solution $\vec{\phi}_0$ using (7) gives the eigenvalue equation

$$\begin{aligned} \left[-\nabla_r^2 - \left(1 - |\vec{\phi}_0(\mathbf{r})|^2\right)\right] \vec{\phi}_1(\mathbf{r}) + 2\left(\vec{\phi}_0(\mathbf{r}) \cdot \vec{\phi}_1(\mathbf{r})\right) \vec{\phi}_0 \\ + \left[k^2 - \omega_k \int d^2 r'\, G_k(\mathbf{r} - \mathbf{r}')\, \vec{\phi}_1(\mathbf{r}')\right] = 0 \ , \end{aligned} \tag{29}$$

where

$$G_k(\mathbf{r} - \mathbf{r}') = \int \frac{d^2 q}{(2\pi)^2}\, \frac{\exp(i\mathbf{q} \cdot \mathbf{r})}{k^2 + q^2} \tag{30}$$

is the Green's function for $(k^2 - \nabla_r^2)$.

For small k, the eigenvalue ω_k can again be calculated using first-order perturbation theory. Rotational invariance ensures, as before, that the unperturbed eigenfunctions $\partial_i \phi_{0j}$ are not mixed by the perturbation. The result can be written in the form

$$\begin{aligned} \omega_k &= k^2\, \frac{\int d^2 r\, (\nabla \vec{\phi}_0)^2}{\int d^2 r \int d^2 r'\, G_k(|\mathbf{r} - \mathbf{r}'|)\, \nabla \vec{\phi}_0(\mathbf{r}) \cdot \nabla \vec{\phi}_0(\mathbf{r}')} \\ &= k^2\, \frac{\int d^2 q\, q^2\, |\vec{\phi}_0(\mathbf{q})|^2}{\int d^2 q\, [q^2/(k^2 + q^2)]\, |\vec{\phi}_0(\mathbf{q})|^2} \ . \end{aligned} \tag{31}$$

The Fourier-space form is more convenient for present purposes. From the result $\vec{\phi}_0 \to \hat{\mathbf{r}}$ for $r \gg \xi$ it follows that $|\vec{\phi}_0(\mathbf{q})|^2 \to (2\pi)^2/q^4$ for $q\xi \ll 1$. Using this result for all q, and introducing ultraviolet and infrared cutoffs $1/\xi$ and $1/R$ respectively as required, the result for ω_k can be written as

$$\begin{aligned} \omega_k &= k^2\, \frac{\int_{1/R}^{1/\xi} q\, dq/q^2}{\int_{1/R}^{\infty} q\, dq/[q^2(k^2 + q^2)]} \\ &= k^4\, \frac{\ln(R/\xi)}{\ln(kR)} \end{aligned} \tag{32}$$

to leading logarithmic accuracy.

One again this result requires careful interpretation. Taking the limit $R \to \infty$ at fixed k gives $\omega_k = k^4$, suggesting the coarsening growth law $L(t) \sim t^{1/4}$. The R-dependence enters (32), however, from the requirement to cut off infrared divergences associated with the far field of the vortex. In the phase-ordering context, the far field is cut off at scale L by other vortex lines. This means we should replace R by L in (32) (see the parallel discussion after (17)). With $k \sim 1/L$ and $\omega_k \sim 1/t$ as usual, this leads to the coarsening growth law $L(t) \sim (t \ln t)^{1/4}$, in perfect agreement with the energy-scaling result [3,4].

The extension to intermediate dynamics, controlled by an exponent μ, is again straightforward. We simply replace $(k^2 + q^2)$ by $(k^2 + q^2)^{\mu/2}$ in (32), with the result

$$\omega_k = k^{2+\mu} \frac{\ln(R/\xi)}{\ln(kR)} \ , \tag{33}$$

to leading logarithmic accuracy, for any $\mu > 0$. The coarsening growth law becomes $L \sim (t \ln t)^{1/(2+\mu)}$.

Finally, long-range interactions can be included. The expression for ω_k combines the numerator from (15) with the denominator from (31). Replacing $(k^2 + q^2)$ by $(k^2 + q^2)^{\mu/2}$ in the denominator, and using (16) for the numerator, gives the result for general μ and $\sigma < 2$:

$$\omega_k \sim k^{\sigma+\mu} \ . \tag{34}$$

Strictly the numerator and denominator generate factors $\ln(\alpha k R)$ and $\ln(\beta k R)$ (with $\alpha \neq \beta$ in general) which we have cancelled in (34). The cancellation is strictly valid in the limit $R \to \infty$ at fixed k. In the coarsening context, where $k \sim 1/L$ and $R \sim L$, these factors are of order unity so (34) still holds, to give $z = \sigma + \mu$.

IV. CONCLUSION

We have discussed a general method, based on the relaxation of a sinusoidally perturbed topological defect (e.g. a domain wall or vortex line), for inferring growth laws in phase-ordering systems. The underlying assumption is that there is a single characteristic length scale ('dynamical scaling') so that defect relaxation is either the sole or dominant coarsening mechanism, or occurs at the same rate as the underlying coarsening process. If this assumption does not hold, there need be no connection between the relaxation spectrum and the coarsening exponent. An example where the present approach fails has recently been given [12].

In this paper we have considered only systems with purely dissipative dynamics. These systems can also be addressed using energy-scaling arguments, which involve equating two independent estimates of the energy dissipation rate [3,4]. In all cases, the results from the two different approaches agree. As yet, however, it has not proved possible to extend the energy-scaling technique beyond purely dissipative systems. The defect relaxation method, however, does not suffer from this limitation. Indeed, Shinozaki [13] has studied the interfacial relaxation spectrum in an incompressible binary fluid, including hydrodynamic effects, and has obtained $\omega_k \sim k$ for $k \to 0$, consistent with the linear growth, $L(t) \sim t$, of the coarsening scale predicted by Siggia [14] for bicontinuous phases.

Another recent application of this approach is to the coarsening of systems which exhibit lamellar structures in equilibrium, such as Rayleigh-Bénard convective rolls as described by the Swift-Hohenberg equation, or block copolymers in the weak segregation regime [15]. The dynamics of these systems has attracted much recent attention [16].

In the coarsening regime the lamellae do not form parallel stripes but rather exhibit a globally isotropic, but locally striped, structure with a characteristic length scale (as measured, for example, by the typical radius of curvature of the stripes) which increases with time, and gives the length scale over which the stripes are locally roughly parallel. The stripe pattern itself has, in the two-dimensional systems studied numerically, topological defects in the form of disclinations. It is not yet clear whether the coarsening is described by a single growing length scale in these systems, as different measures of this scale give different results [16]. An analysis of the dynamics of a modulated lamellar structure along the lines presented here gives the relaxation rate $\omega_k = (\epsilon/256)k^2 + k^4$, where ϵ is measure of the quench depth [15]. For shallow quenches, therefore, one has $\omega_k \simeq k^4$, suggesting a $t^{1/4}$ growth at not-too-late times, whereas the growth rate inferred from the evolution of the structure factor is closer to $t^{1/5}$ [16]. The disclinations, however, have not been included in this approach, and it is possible that these coarsen more slowly than the interfaces. More work is needed to clarify this point.

In summary, a study of defect relaxation provides a simple way to determine growth exponents in coarsening systems. This approach should be reliable when the coarsening is described by a single characteristic length scale. In the cases studied here, the results obtained are identical to those derived using the energy-scaling method. While the underlying assumptions (dynamical scaling) are the same for both methods, the ideas involved in the present approach are rather simpler, and the method is not restricted to purely dissipative dynamics. The energy-scaling method, on the other hand, is not restricted to extended defects, i.e. it can be used for systems with point defects or, indeed, no defects at all.

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[1] See A. J. Bray, Adv. Phys. **43**, 357 (1994) for a recent review.
[2] I. M. Lifshitz and V. V. Slyozov, J. Phys. Chem. Solids, **19**, 35 (1961); C. Wagner, Z. Elektrochem **65**, 581 (1961).
[3] A. J. Bray and A. D. Rutenberg, Phys. Rev. E **49**, R27 (1994).
[4] A. D. Rutenberg and A. J. Bray, Phys. Rev. E **51**, 5499 (1995).
[5] P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).
[6] J. S. Langer and L. A. Turski, Acta. Metall. **25**, 1113 (1977).
[7] D. Jasnow and R. K. P. Zia, Phys. Rev. A **36**, 2243 (1987).
[8] A. Shinozaki and Y Oono, Phys. Rev. E **47**, 804 (1993).
[9] A. J. Bray, Phys. Rev. E **47**, 3191 (1993).
[10] A. J. Bray, Phys. Rev. B **41**, 6724 (1990).
[11] A. Onuki, Prog. Theor. Phys. **74**, 1155 (1985).
[12] M. Siegert, M. Plischke, and R. K. P. Zia, Phys. Rev. Lett. **78**, 3705 (1997).
[13] A. Shinozaki, Phys. Rev. E **48**, 1984 (1993).
[14] E. D. Siggia, Phys. Rev. A **20**, 595 (1979).
[15] J. J. Christensen and A. J. Bray, unpublished.
[16] See, e.g. K. R. Elder, J. Viñals, and M. Grant, Phys. Rev. A **46**, 7618 (1992); Phys. Rev. Lett. **68**, 3024 (1992); J. J. Christensen, K. Elder, and H. C. Fogedby, Phys. Rev. E **54**, R2212 (1996).